Turbulent spectrum of the earth's ozone field

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Abstract

The Total Ozone Mapping Spectrometer (TOMS) database is subjected to an analysis in terms of the Karhunen-Loeve (KL) empirical eigenfunctions. The concentration variance spectrum is transformed into a wavenumber spectrum, $E_c(k)$. In terms of wavenumber $E_c(k)$ is shown to be $O(k^{-2/3})$ in the inverse cascade regime, $O(k^{-2})$ in the enstrophy cascade regime with the spectral *knee* at the wavenumber of barotropic instability.The spectrum is related to known geophysical phenomena and shown to be consistent with physical dimensional reasoning for the problem. The appropriate Reynolds number for the phenomena is $Re \approx 10^{10}$.

Atmospheric mixing is effected at horizontal scales which are large compared with the scale height ($\approx 10km$), which with inhibition of vertical motion by planetary rotation and stable stratification contributes to the two dimensional picture of atmospheric activity.[1,2] In addition to its essential role in meteorology, current interest in mixing is enhanced by its role in regard to the behavior of the Antarctica ozone *hole*[3] and to the lack of such an *effect* in the Arctic.[4]

Our investigation is based on satellite records of the earth's ozone fields. We have analyzed fifteen years of daily ozone fields of the TOMS (Total Ozone Mapping Spectrometer) database.[5] Due to technical and natural causes each daily record contains gaps in the form of missing pixels. A typical snapshot appears in Figure 1. The dark regions represent areas of missing pixels caused partly by technical failure and in part due to polar night (measurements are based on reflected light). Each record is *stitched* together from sixteen separate records obtained from south-north synchronous orbits that are taken in a twenty four hour period from the satellite, Nimbus.

The availability of such a large data set recommends statistical analyses, and we focus on spectral properties of the ozone field. Although ozone production (equatorial regions) and depletion (polar regions) result from complex chemical reactions[6] these represent relatively weak sources and sinks and we follow common practice and regard ozone as a passive scalar. The variance spectrum of a scalar contaminant in turbulent flows has been recently reviewed by Sreenivasan.[7] Functional estimates for the concentration variance spectrum (in homogeneous isotropic turbulence) follow from dimensional arguments based on those first given by Kolmogorov leading to the famous $E_K(k) = K\epsilon^{2/3}k^{-5/3}$ energy spectrum for the inertial range.[8]

Both Obukhov and Corrsin[9] show that an inertial range can exist, in particular the variance per wavenumber of concentration, c, denoted by $E_c(k)$, has the form,

$$E_c(k) = C\chi\epsilon^{-1/3}k^{-5/3}, \tag{1}$$

where C is a dimensionless constant, ϵ is the usual turbulent energy transport rate, and $\chi =< \kappa(\nabla c)^2 >$ is the appropriate *dissipation* rate. Thus the concentration spectrum appears tied to the corresponding velocity spectrum. A review of experimental observations[7] shows departures from the universal form (1), except perhaps at very high Reynolds numbers. For relatively small diffusional effects Batchelor[10] has shown that $E_c = O\left(k^{-1}\right)$. This has been shown to hold under less restrictive hypothesis.[11,12] Except for a recent simulation[12] confirmation of this result has been elusive. Predictions of anomolous scalings have also appeared.[13]

Arguments leading to the above spectra are unaltered when applied to two dimensional turbulence. However, the interpretation of the cascade of energy represented by the spectrum $E_K(k)$ requires some additional remarks. Both energy E, and enstrophy $\Omega = \int (\nabla \wedge \mathbf{u})^2 d\mathbf{x}$, are inviscid invariants in two dimensions. As a result of this Kraichnan and Batchelor[14] have shown that the Kolmogorov spectrum, E_K, represents an inverse cascade of energy from smaller to larger scales, and that there also exists a second cascade from small k to large k given by the enstrophy $\Omega(k) = C_o\chi_o k^{-1}$(with log correction) and hence an energy spectrum $E(k) \propto k^{-3}$, where C_o is a dimensionless constant and $\chi_0 = \nu\overline{(\nabla\omega)^2}$ (also see[15]). Support for $E = O(k^{-3})$ comes from many direct simulations.[16] However, recent very large scale simulations show substantial divergence in the $O(k^{-5/3})$ inverse cascade range.[17] Observational data from the atmosphere is not definitive and although a power law energy spectrum is indicated in the enstrophy range the exponent appears to lie between -2 and -3.[18] In particular, Schoeberl and Bacmeister[18] suggest that the exponent is -2 down to scales in the 10 kilometer range!

A difficulty in interpreting these results for E_c already appears. Focusing on k large, it might be supposed in

analogy with three dimensions, that $E_c\ (k) = O\left(k^{-3}\right)$ i.e., it should follow the energy spectrum. On the other hand, the vorticity (a scalar) formally satisfies the same convection equation as does a passive scalar, and from this one might suppose that $E_c = O\left(k^{-1}\right)$, the Batchelor spectrum. As will be seen shortly, neither of these holds for E_c in the atmosphere.

Other possible scalings for E_c have appeared in the literature. For quasi two-dimensional turbulence Falkovich and Medvedev[19] find $E = O\left(k^{-7/3}\right)$ for large k. Saffman,[20] in considering the Burgers equation,[21] observed that its solutions are nearly piecewise discontinuous which leads to $E_c = O\left(k^{-2}\right)$. Pierrehumbert using concepts from chaotic mixing has obtained a variety of scalings for E_c from both mathematical and physical models.[22]

Satellite images (see Fig.1) are clearly inhomogeneous and a transformation to wavenumber concepts is required. As will be seen the Karhunen-Loeve(KL) procedure[23] is ideally suited for this purpose. In particular, the snapshot method[24] considerably reduces the needed computation effort. However, the presence of gappy data required modification of the methodology.[25] This, as well as an extensive analysis of the results, appears in Manin et al[26] and a mathematical treatment is also given elsewhere.[27]

To connect the usual wavenumber spectrum with that obtained from the empirical eigenfunctions we recall an earlier discussion.[28] The concentration fluctuation of ozone is denoted by $c(\mathbf{x}, t)$. For purposes of later dimensional reasoning we write the dimensions of c as $\dim[c] = m/l^2$, where m refers to molecules (of ozone) per area since the data are two dimensional. The mean variance in the homogeneous case is given by

$$\overline{c^2} = \frac{1}{A}\int c^2 d\mathbf{x} = \int \mathcal{E}_c(\mathbf{k}) d\mathbf{k} = \int E_c(k) dk. \tag{2}$$

Thus $\dim[\mathcal{E}_c] = m^2 l^{-2}$ and $\dim[E_c] = m^2 l^{-3}$. To treat the inhomogeneous case corresponding to the data we consider the correlation

$$K_c(\mathbf{x}, \mathbf{y}) = < c(\mathbf{x}) c(\mathbf{y}) > \tag{3}$$

which from KL can be written in spectral form,

$$K_c = \sum_n \lambda_n \psi_n(\mathbf{x}) \psi_n(\mathbf{y}) \tag{4}$$

where $\{\psi_n\}$ are the eigenfunctions of the operator K_c. The total variance is given by

$$< \int c^2(\mathbf{x}) d\mathbf{x} >= TrK = \sum_n \lambda_n. \tag{5}$$

An eigenvalue λ_n represents the average variance allocated to the projection of c onto ψ_n. The summation (5) is the natural generalization of (2) to the inhomogeneous case. Each λ_n represents the variance in a state, thus generalizing $\mathcal{E}_c(\mathbf{k})$ and has the same dimensions,

$$\dim[\lambda] = \dim[\mathcal{E}] = m^2 l^{-2}. \tag{6}$$

In Figure 2 we display in doubly logarithmic form λ_n versus index n. As is seen the variance spectrum falls, to good approximation, on two different power laws

$$\lambda_n \propto \begin{cases} n^{\alpha_i};\ n < 35, \quad \alpha_i = .85 \pm .035 \\ n^{\alpha_o};\ n > 50, \quad \alpha_o = -1.56 \pm .022 \end{cases} \tag{7}$$

The error bounds appearing in (7) and are based only on the least squares fit to the data, and not on the methods used in arriving at the spectrum which appears in Figure 2. The region of the *knee,* $35 \le n \le 50$, will be discussed below.

In order to relate n to k we observe that in the homogeneous case modes carrying variances larger than those at k can be counted in number, N, as

$$N \propto k^2, \text{ whence } k \propto N^{1/2}. \tag{8}$$

Before continuing this reasoning it is instructive to verify the accuracy and content of these relations. For this purpose we employ the inverse relation between wavenumber and length scale. Thus (8) implies that the length scale, L_n, of the n^{th} eigenfunction bears the following dependence on the index

$$L_n \propto n^{-1/2}. \tag{9}$$

An informal perusal of the eigenfunctions themselves supports this relationship between characteristic length and index. To quantify this we have computed the correlation length of each eigenfunction[26] and the result is plotted in Figure 3. It is clear from this figure that (9) provides an excellent fit to the data in the two asymptotic regimes. The region of the *knee* is the only anomaly and it appears as a plateau in the figure and corresponds to just one scale.

$$2\pi/k_* = L_* \approx 4000 \text{ km}. \tag{10}$$

It is generally stated in the geophysical literature[29,1] that the baroclinic instability gives rise to a pattern. of wavenumber roughly seven. I.e., the unstable pattern is made up of approximately seven pairs of cyclonic/anticyclonic motions. With some indulgence on the part of the viewer, eigenfunction ψ_{44} shown in Figure 4 appears to have this property. Roughly speaking, each of the eigenfunctions in the range $35 \le n \le 50$ shows this spatial arrangement. To explain the plateau in Figure 3 we

suggest an analogy with the von Karman vortex trail. In that case a period seven disturbance requires fourteen independent modes for its description.[30] In view of the nature of the results this would appear to be a reasonable explanation.

The energetics of the atmosphere has its origin in solar heating. However, dynamical activity introduces L_* as the lengthscale at which mechanical energy is supplied, and is thus the significant lengthscale of the problem. If ϵ is the energy transport rate which characterizes the inverse energy cascade, then

$$\chi_0 = k_*^2 \epsilon \tag{11}$$

characterizes the enstrophy cascade to higher wavenumbers. Using estimates for the physical parameters of the atmosphere and L_* from (10) we obtain the Reynolds number, $R \approx 10^{10}$.

We now return to the implications of the power laws for λ_n, (7) to the wavenumber spectrum. In keeping with customary practice we consider the variance per wavenumber $E_c(k) = k\mathcal{E}_c(k)$. It follows from (7) and (8) that

$$E_c(k) \propto \begin{cases} k^{-2/3}, & k < k_* \\ k^{-2}, & k > k_* \end{cases} \tag{12}$$

The more precise exponents are entered for suggestive reasons. The high wavenumber exponent lies slightly outside the error bound (However, it should be noted that Kraichnan[14] actually estimated the enstrophy fall-off as $1/k^3 ln^{1/3} k$, see also[15]). With the exception of the Saffman-Burgers spectrum,[20] theoretical predictions lie outside the above range.

In view of the relationship (12), we now consider the consequences of elementary dimensional analysis. A straightforward argument yields,

$$E_c(k) = \chi \epsilon^{-1/3} k_*^{-5/3} f(k/k_*) \tag{13}$$

Equation (twe) implies

$$f(k/k_*) \sim \begin{cases} c_i (k/k_*)^{-2/3}, & k/k_* \ll 1 \\ c_o (k_*/k)^2, & k/k_* \gg 1 \end{cases} \tag{14}$$

where c_i and c_o are dimensionless constants. In these terms

$$E_c(k) \sim \begin{cases} c_i \chi \epsilon^{-1/3} k_*^{-1} k^{-2/3} & k/k_* \ll 1 \\ c_o \chi_o^{-1/3} \chi k_* k^{-2} & k/k_* \gg 1 \end{cases} \tag{15}$$

where the first form is appropriate for the inverse energy cascade and the second form is in a form appropriate for the enstrophy cascade. It should be noted that k_* (or L_*) does not disappear under either asymptotic limit. In fact as the functional form (13) implies it would be impossible to eliminate this parameter entirely in both limits.

The spectra obtained above covers a wide range. The range $k/k_* < 1$ extends to the 10,000 *km* wavelength limit imposed by dynamics[31,1] and the $k/k_* > 1$ extends down to wavelengths of the order of 100*km*, the resolution of the data. Smith and Yakhot[32] have recently suggested that an inverse cascade for $E(k)$ exists for a range of wavelengths greater than 10 *km* (but see Schoeberl and Bacmeister[18]). This is based on the assertion that cumulus cloud activity acts as an energy source. We see no evidence for this but do not regard this as a contradiction since the resolution of the satellite data is greater than 100 *km*. It is of course vexing that much the theory and numerical experiments (also[33]) discussed earlier do not agree with the observed satellite analysis which we present above. Only the Saffman-Burgers spectrum shows agreement. To test this further we have looked at the 'discontinuity' patterns of the data and find that $|\triangledown c|^2$ shows filamentous one dimensional patterns .[26]A possible explanation for such strand-like patterns has been discussed recently. Both satellite observations and computer simulations show the presence of tongues of stratospheric air extending from the tropics to mid-latitudes. These result from the breaking of Rossby waves at the edge of the polar vortices.[34]

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